V $\mathbb{C}$-ev. $\dim_{\mathbb{C}} V = n \in \mathbb{N}$

$f: V \times V \longrightarrow \mathbb{C}$ sesqui-simétrico $f(u,v) = \overline{f(v,u)}$

Obs: $\beta = (w_i)_i$ base de V

$[f]_\beta = \left(f(w_i, w_j)\right)_i^j \in \mathbb{C}^{n \times n}$; $u = \sum_i \mu_i w_i$

$v = \sum_i \nu_i w_i$

$$f(u,v) = \overline{[u]}^t \cdot [f] \cdot [v]$$

$$= (\bar{\mu}_1, \ldots, \bar{\mu}_n) \begin{pmatrix} & \cdots & \\ & f(w_i, w_j) & \\ & \cdots & \end{pmatrix} \begin{pmatrix} \nu_1 \\ \vdots \\ \nu_n \end{pmatrix}$$

Obs 2: V $\mathbb{C}$-ev.

$\bar{V}$ $\mathbb{C}$-ev, tal que $V = \bar{V}$ como conjunto, com a operação soma.

$v \in V$ $\quad \lambda v \in V$

$v \in \bar{V}$ $\quad \lambda \cdot v = \bar{\lambda} v$

i) $\varphi: V \longrightarrow W$ conjugado linear $\quad \varphi(u+v) = \varphi(u) + \varphi(v)$

$\bar{\mathbb{C}}$-linear $\quad \varphi(\lambda u) = \bar{\lambda} \varphi(u)$

$\varphi: \bar{V} \longrightarrow W$ $\mathbb{C}$-linear

ii) $f: V \times W \longrightarrow \mathbb{C}$ sesqui-bilinear

$f: \bar{V} \times W \longrightarrow \mathbb{C}$ bi-linear

$[F] = \overline{[F]}^t$ Sesqui-simétrico (ou Hermitiano)

$[F] = [F]^t$ Simétrico

Teo. de Classificação de Formas Sesqui-Simétricas sobre $\mathbb{C}$

$F: V \times V \to \mathbb{C}$ sesquisimétrico

$\exists\, \beta$ base de V ($\mathbb{C}$-ev. $\dim_{\mathbb{C}} V = n < +\infty$)

$$[F]_\beta = \left(\begin{array}{c|c} Id_r & \\ \hline & 0 \end{array}\right)$$

r é unico

Prova ✓ (existência)

Unicidade $[F]_{\beta'} = \left(\begin{array}{c|c} Id_{r'} & \\ \hline & 0 \end{array}\right)$

$$[F]_{\beta'} = \overline{C_{\beta'\beta}}^t \, [F]_\beta \, C_{\beta'\beta}$$

$\in GL_{\mathbb{C}}(n)$

Not: $\dagger$ adjunta

• Posto de $[F]_{\beta'}$ e $[F]_\beta$ é idêntico.

r é o posto da $[F]_\beta$ ∎

Teo. Classificação Formas Simétricas sobre $\mathbb{R}$

$f: V \times V \to \mathbb{R}$

(...) Forma canónica

$\exists\, \beta$ base de V

$\beta = (v_1, \dots, v_n)$

$$[F]_\beta = \begin{pmatrix} Id_{r_+} & & \\ & -Id_{r_-} & \\ & & 0 \end{pmatrix}$$

únicos r_+, r_-

Prova da existência ✓

Unicidade ...

$L_0 = \{ u \in V : f(u, \cdot) \equiv 0 \} = \langle v_{r_+ + r_- + 1}, \dots, v_n \rangle$

↑ Afirmo

$L_+ = \langle v_1, \dots, v_{r_+} \rangle$

$L_- = \langle v_{r_+ + 1}, \dots, v_{r_+ + r_-} \rangle$

i) $F(u_+, u_+) \geq 0 \quad \forall\, u_+ \in L_+$

$= 0$ se e só se $u_+ = 0$

$u_+ = \mu_1 v_1 + \dots + \mu_{r_+} v_{r_+}$ (...)

$$F(u_+, u_+) = (\mu_1, \dots, \mu_r) \, {+Id} \begin{pmatrix} \mu_1 \\ \vdots \\ \mu_r \end{pmatrix} = +\|(\mu_1, \dots, \mu_{r_+})\|^2$$

Definição:

$f_1 : V \times V \to \mathbb{R}$

$f_2 : W \times W \to \mathbb{R}$

(V, f_1) e (W, f_2) são isométricas se existe $\tau: V \to W$ isomorfismo linear tal que

$$V \times V \xrightarrow{f_1} \mathbb{R}, \quad (\tau, \tau): V \times V \to W \times W, \quad W \times W \xrightarrow{f_2} \mathbb{R}$$

$$f_2(\tau(v), \tau(w)) = f_1(v, w)$$

$\Updownarrow$

β_V base de V, β_W base de W

$$[f_1]_{\beta_V} = [\tau]_{\beta_V \beta_W}^t \, [f_2]_{\beta_W} \, [\tau]_{\beta_V \beta_W}$$

Vamos a provar que (r_+, r_-) "assinatura da f" é única.

$$[F]_{\beta'} = \begin{pmatrix} Id_{r'_+} & & \\ & -Id_{r'_-} & \\ & & 0 \end{pmatrix}$$

Estamos comparando (r_+, r_-) e (r'_+, r'_-)

Temos

$$V = (L_0 \oplus L_+) \oplus (L_-)$$
$$= (L_0 \oplus L'_+) \oplus (L'_-)$$

$V = A \oplus B = A' \oplus B'$

$\dim A > \dim A'$

Podemos definir projeção em cada fator

$v \in V \qquad v = a' + b' \qquad$ onde a, b são únicos

$v \mapsto a' \qquad$ está bem definida

$A \overset{i}{\hookrightarrow} V \overset{P}{\longrightarrow} A'$

inclusão projeção

A composição é linear $A \longrightarrow A'$ $\Rightarrow$ tem núcleo

Obs: $\exists\, v \in A \cap B'$ não nulo.

No caso $V = \overbrace{L_0 \oplus L_+}^{B} \oplus \overbrace{L_-}^{A}$

$L_0 \oplus L_+' \oplus L_-'$

Temos que $r_- \neq r_-'$ podemos supor $r_- > r_-'$

$\exists\, 0 \neq v \in L_- \cap (L_+' \oplus L_0)$

$F(v, v) < 0$

$F(v, v) \geq 0 \qquad v \in L_+' \oplus L_0$

$v = u_+ + u_0$

$\underbrace{F(u_+, u_+)}_{\geq 0} + 2F(u_+, u_0) + F(u_0, u_0)$

$F: \mathbb{R}^2 \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ associada a $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

obs: $\det(\cdot) = -1 \neq 0$ ⇒ forma F é não-degenerada

$$F(e_1, e_1) = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 0$$

$F(e_1, \cdot)$ é um funcional não nulo.

$F|: \mathbb{R}e_1 \times \mathbb{R}e_1 \longrightarrow \mathbb{R}$ é degenerada

<u>Ex</u>: assinatura $(F) = (1, 1)$ ou seja forma canônica associada

$$[F]_\beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Algoritmo de Ortogonalização

$f: V \times V \to \mathbb{R}$ ($\mathbb{C}$) simétrica (sesquilinear)

$S_1 \subseteq S_2 \subseteq \dots \subseteq S_i \subseteq V$ subespaços

$f|: S_\ell \times S_\ell \to \mathbb{R}$ ($\mathbb{C}$) não degenerada

$\dim S_\ell = \ell$

(Por exemplo, dados $u_1, \dots, u_i$ seq. de vetores l.i., temos $S_\ell = \langle u_1, \dots, u_\ell \rangle$)

Vamos a construir seq. $\tilde{u}_1, \dots, \tilde{u}_\ell$

$S_\ell = \langle \tilde{u}_1, \dots, \tilde{u}_\ell \rangle$

$f|: S_\ell \times S_\ell \to \mathbb{R}$ ($\mathbb{C}$) $\quad [f|]_{(\tilde{u}_1, \dots, \tilde{u}_\ell)} = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_\ell \end{pmatrix}$

cada $\lambda_j \neq 0$.

Algoritmo: $\tilde{u}_1 = u_1$

$\tilde{u}_2 = u_2 - \lambda u_1$

Determinar λ para que $f(\tilde{u}_1, \tilde{u}_2) = 0$

$0 = f(\tilde{u}_1, \tilde{u}_2) = f(u_1, u_2) - \lambda \underbrace{f(u_1, u_1)}_{\neq 0}$

$\lambda = f(u_1, u_2) / f(u_1, u_1)$

$\vdots$

Temos $\tilde{u}_1, \dots, \tilde{u}_{j-1}$ $\quad f(\tilde{u}_k, \tilde{u}_\ell) = 0 \quad k \neq \ell \leq j-1$

$f(\tilde{u}_k, \tilde{u}_k) = \lambda_k \neq 0$

obs: u_j é l.i. com

$\tilde{u}_j = u_j - \sum_{\ell=1}^{j-1} \mu_\ell \tilde{u}_\ell$

$0 = f(\tilde{u}_k, \tilde{u}_j)$ para $k \leq j-1$

$= f(\tilde{u}_k, u_j) + f(\tilde{u}_k, -\sum_{\ell=1}^{j-1} \mu_\ell \tilde{u}_\ell)$ } Linearidade na segunda coord.

$= \underbrace{f(\tilde{u}_k, u_j)}_{\checkmark} - \sum_{\ell=1}^{j-1} \underbrace{\mu_\ell}_{?} \underbrace{f(\tilde{u}_k, \tilde{u}_\ell)}_{\substack{=0 \;\; k \neq \ell \\ =\lambda_k \;\; k=\ell}}$

$0 = f(\tilde{u}_k, u_j) - \mu_k \lambda_k$

$\mu_k = f(\tilde{u}_k, u_j) / \lambda_k$

$\beta = (u_1, \dots, u_n) \rightsquigarrow \tilde{\beta} = (\tilde{u}_1, \dots, \tilde{u}_n)$ $\quad$ (Pensar $V = S_n$)

$C_{\tilde{\beta}\beta} \cdot e_i = [\tilde{u}_i]_\beta = \begin{pmatrix} * \\ \hline 0 \end{pmatrix}$ i-ésima

$$C_{\tilde{\beta}\beta} = \begin{pmatrix} \diagdown & * \\ 0 & \diagdown \end{pmatrix}$$

Critério dos Menores

$A \in \mathbb{K}^{n \times n}$ $\quad A = (a_{ij})_{1 \leq i \leq n}^{1 \leq j \leq n} =$

$A_d = (a_{ij})_{i \leq d}^{j \leq d}$

Dada $A \in \mathbb{R}^{n \times n}$, forma: $\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ simétrica

Supor $\det(A_d) \neq 0 \quad \forall d \leq n$

(obs: equivale $\langle e_1, \dots, e_i \rangle = S_i$; A forma é não-deg. em S_i)

$a_d = \det(A_d)$

Computar:

$a_1, \; a_2/a_1, \dots, a_n/a_{n-1}$

r_+: índice de positividade corresponde à quantidade de valores > 0 da lista anterior.

< 0

$\mathcal{E} = (e_1, \dots, e_n)$

$S_k = \langle e_1, \dots, e_k \rangle$

$[F]_{\mathcal{E}} = A$

$F|_k : S_k \times S_k \longrightarrow \mathbb{R}$

$[F|_k]_{\mathcal{E}} = A_k$

Aplicando Alg. Gram-Schmidt $\tilde{\mathcal{E}} = (\tilde{e}_1, \dots, \tilde{e}_n)$

$$C_{\tilde{\mathcal{E}}\mathcal{E}} = \begin{pmatrix} \diagdown & * \\ 0 & \diagdown \end{pmatrix}$$

$$[F]_{\tilde{\mathcal{E}}} = \begin{pmatrix} \pm 1 & & & \\ & \pm 1 & & \\ & & \ddots & \\ & & & \pm 1 \end{pmatrix} = C_{\tilde{\mathcal{E}}\mathcal{E}}^{t} \, A \, C_{\tilde{\mathcal{E}}\mathcal{E}}$$

Not: $C := C_{\tilde{\mathcal{E}}\mathcal{E}}$ $\quad C = \left(\begin{array}{c|c} C_k & * \\ \hline 0 & * \end{array}\right)$ $\quad C^t = \left(\begin{array}{c|c} C_k^t & 0 \\ \hline * & * \end{array}\right)$

$$\begin{pmatrix} C_k^t & 0 \\ * & * \end{pmatrix} \begin{pmatrix} A_k & * \\ * & * \end{pmatrix} \begin{pmatrix} C_k & * \\ 0 & * \end{pmatrix} =$$

$$\begin{pmatrix} C_k^t \cdot A_k & * \\ * & * \end{pmatrix} \begin{pmatrix} C_k & * \\ 0 & * \end{pmatrix} =$$

$$\begin{pmatrix} C_k^t A_k C_k & * \\ * & * \end{pmatrix}$$

$$\underbrace{\pm 1}_{\text{sinal}} = \det(C_k^t A_k C_k) = \underbrace{\det{}^2(C_k)}_{>0} \underbrace{\det(A_k)}_{\text{coincide com este sinal.}}$$

$\blacksquare$